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DISJOINT PATHS IN A RECTILINEAR GRID

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Dedicated to Tibor Gallai on his seventieth birthday

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We give a good characterization and a good algorithm for a special case of the integral multicommodity flow problem when the graph is defined by a rectangle on a rectilinear grid. The problem was raised by engineers motivated by some basic questions of constructing printed circuit boards.

1. Introduction

One of the central topics in graph theory concerns the existence of disjoint paths under various constraints. Basic results are due to Gallai [3, 4], Tutte [13], Menger [10], König [6], Mader [9], Seymour [12], Lovász [8]. In the present paper we discuss the following problem in this field.

Given an undirected graph, find k edge-disjoint paths between k pairs of vertices specified in advance. This problem, often called the *disjoint paths problem*, is a specialization of the integral multicommodity flow problem which belongs to the hard class of NP-complete problems even if $k=2$ [5].

For the disjoint paths problem there is a good characterization, due to Seymour [12], for $k=2$, but the question is open for any fixed $k \geqq 3$ [5]. However, the following special case, related to wiring problems of printed circuit boards, can be well-characterized. The proof also provides a good algorithm.

In a rectilinear grid (or plane lattice) we are given a closed rectangle T (bounded by lattice lines) and k pairs of distinct lattice points of the boundary. T defines a finite subgraph G of the plane grid in the natural way (which has $n \cdot m$ vertices when m horizontal and n vertical lines intersect T). The purpose of this paper is to give an answer to the disjoint paths problem in this special case.

The problem was formulated by I. Abos, an electrical engineer who (with his co-workers) worked out a general computer program for designing printed circuits boards [1].

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It should be noted that the engineering literature is quite rich in works concerning such designing questions and a large number of models, approaches, and algorithms are offered. However, the great part of these procedures uses heuristics and a typical conclusion of such a paper is that "the computer program was able to construct 90—95 per cent of the wirings". Of course, such programs may work quite well in practice, but mathematically well-founded models may help to reach further improvements.

In the mathematical context our problem is strongly related to that of Okamura and Seymour [11]. The precise relationship will be discussed in Section 4.

In the first part of the paper we consider the even more special problem when one member of each pair is on the upper horizontal line while the other one is on the lower line. We refer to this case as *bipartite*. The non-bipartite case, when the terminals are allowed to be arbitrarily positioned on the boundary of T, will be presented in Section 3.

By a *column (row)* of T we mean a region in T between two consecutive vertical (horizontal) lattice lines. The *path congestion* (or *congestion*) of a column c is the number of terminal pairs separated by c. We call a lattice point *exposed* if it is not a terminal. An obvious necessary condition for a solution is the *column criterion*: the congestion of any column is at most m, the number of horizontal lines.

2. The bipartite case

Unfortunately the column criterion is not sufficient in general, as each of the next examples shows.

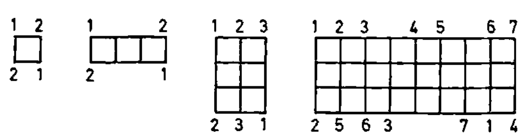

Fig. 1

Surprisingly, a small restriction makes the column criterion sufficient. Namely, we have the following result.

Theorem 1. *If at least one corner point of T is not a terminal vertex, the column criterion is necessary and sufficient for the existence of edge-disjoint paths between the corresponding terminals.*

Proof. We prove the suffiency. Suppose that the four corners of T are $A=(1, m)$, $B=(n, m)$, $C=(m, 1)$, $D=(1, 1)$. For a terminal pair i let $U(i)$ and $L(i)$ denote the x-coordinates of the upper and lower member of i, respectively. We shall adopt the notational convention that the x-coordinate of a point P will be denoted by the same term P.

It may be assumed that m is equal to the maximal congestion Δ. We proceed by induction on m. The case $m=1$ being trivial, let $m>1$. Assume that point A (the left upper corner of T) is exposed.

We can assume that $L(i)\neq U(i)$, that is no *trivial path* exists. For otherwise, join them by a vertical path, then remove this lattice line from the rectangle. In the reduced problem the column criterion holds again and A is also exposed.

The algorithm will consist of m phases. During each phase, both the number of horizontal lines and the maximal congestion will be reduced by one. We describe and analyse only the first phase. This will result in a sequence $U'(i)$ $(i=1, 2, \dots, k)$ of distinct points which tells path i where to go to on the upper line. In particular, $U'(i)=U(i)$ means that path i will start vertically down.

In order to have a valid algorithm and proof, the sequence $U'(i)$ $(i=1, \dots, k)$ must be determined in such a way that the induction hypothesis should hold for the terminal pairs $U'(i)$, $L(i)$ $(i=1, 2, \dots, k)$ on the rectangle T' consisting of one less horizontal lattice line (i.e. the corners of T' are $A'=(1, m-1)$, $B'=(n, m-1)$, $C=(n, 1)$, $D=(1, 1)$). In other words, the new maximal congestion should be $m-1$ and one corner point of T' must be exposed again.

Our procedure has the feature that the upper corner points are alternately exposed, that is, at the end of the first phase the right upper corner of T' will be exposed, after the second phase the left upper corner will, and so forth.

A path i is said to be a *left-path (right-path)* if $U(i)>L(i)$ $\big(U(i)<L(i)\big)$. Let X, Y be two points on segment AB with $X<Y$. The basic step of our procedure is: "left-pushing" on XY. Assume that X is exposed. An *elementary left-pushing on XY* tells a single left-path how to go on the segment XY. Decide whether there exists a left path i for which $X<U(i)\leqq Y$ and if so, select one, say j, for which $L(j)$ is as small as possible. (If no such path exists, the elementary left-pushing is called *vanishing*.) Set $U'(j)=X$ if $L(j)\leqq X$ and $U'(j)=\max\big(Z\colon X\leqq Z\leqq L(j), (Z, m) \text{ is exposed}\big)$ if $L(j)>X$.

Note that $U'(j)$ is well-defined since X was exposed. Furthermore j becomes trivial for the second phase if $Z=L(j)$ and becomes a right path if $Z<L(j)$. In this latter case the congestion of columns between Z and $L(j)$ has been increased by one. Moreover, by the minimality of j, each point $Z+1, Z+2, \dots, L(j)$ on the upper line is a terminal of a right path.

A *left-pushing on XY* tells several left-paths how to go on the segment XY. First, apply an elementary left-pushing on XY. Assume that path j has been moved to the left. Then the point $X'=U(j)$ on the upper line has become exposed. Apply now an elementary left-pushing on $X'Y$ and repeat this procedure until the actual left-pushing is vanishing.

An *elementary right-pushing* is defined analogously.

The first phase of the algorithm consists of two parts. First, apply a left-pushing on the whole AB. In the second part consider each maximal subsegment XY of AB which has not been covered by any path in the first part and put $Y'=\max\big(Z\colon X\leqq Z<Y, (Z, m) \text{ is exposed}\big)$. (By definition of a left-pushing, X has been exposed and thus Y' does make sense.) For each XY' apply a right-pushing on XY'.

The description of the first phase of the algorithm is now complete. We set $U'(j)=U(j)$ for each path j having not appeared in the left or right-pushing of the first phase.

To prove the correctness of the algorithm, first observe that $U'(i)\neq U'(j)$ when $i\neq j$ and the subpaths which have been defined on AB are edge-disjoint. Furthermore, one can see that, after deleting the trivial paths which have arisen, the right upper corner of the rectangle is exposed. Let us prove now that the maximum congestion in the reduced problem is one less.

The congestion of any column was changed by 0, $+1$ or -1. First, consider those columns whose congestion increased by one. This might occur only in Part 1, if, while accomplishing an elementary left-pushing on XY, we had $L(j)>X$ and $L(j)>Z$. As mentioned, the congestion of each column between Z and $L(j)$ has increased by 1.

Claim 1. *The original congestion of any column between Z and $L(j)$ is at most $\Delta-2$.*

Proof. Since the points $(L(j), m)$ and $(L(j), 1)$ are terminals of a right-path and a left-path, respectively, we have $c(L(j)-1, L(j))=c(L(j), L(j)+1)-2\leqq\Delta-2$ where $c(S, S+1)$ stands for the congestion of the column between S and $S+1$.

We know that all points $Z+1, \dots, L(j)$ on the upper line are terminals of right paths. Therefore $c(Z, Z+1)\leqq c(Z+1, Z+2)\leqq\dots\leqq c(L(j)-1, L(j))\leqq\Delta-2$, as required. ▮

The congestion of any column, which separates the terminals of a left-path, is reduced by 1 in the first part. So we have to deal with the maximal segments XY which have not been covered in the first part. Recall the definition of Y'.

Claim 2. *The (unchanged) congestion of each column between Y' and Y is less than Δ.*

Proof. For $Y'<Z<Y$ each point (Z, m) is a terminal of a right path and thus $c(Y', Y'+1)\leqq c(Y'+1, Y'+2)\leqq\dots\leqq c(Y-1, Y)$. One can see that the point (Y, m) was originally exposed. We have three possibilities for Y.

(i) $Y=B$, that is Y is the right upper corner of T. Then

$$c(Y-1, Y)\leqq 1<m=\Delta,$$

(ii) $(Y, 1)$ is a terminal of a left path in which case

$$c(Y-1, Y)<c(Y, Y+1)\leqq\Delta,$$

(iii) $(Y, 1)$ is not a terminal of a left path and $Y<n$. By Claim 1

$$c(Y, Y+1)\leqq\Delta-2.$$

Since $c(Y-1, Y)\leqq c(Y, Y+1)+1$ the claim follows. ▮

Claim 3. *The new congestion of any column between X and Y' is at most $\Delta-1$.*

Proof. Assume that the column $(Z, Z+1)$ violates the claim $(X\leqq Z<Y')$. Then $(Z, Z+1)$ originally separates Δ terminal pairs of right paths. If anyone of these has an upper terminal $U(i)>X$ then one path has crossed the column $(Z, Z+1)$ in the second part of the algorithm and therefore the new congestion of $(Z, Z+1)$ is less than Δ, contradicting the assumption.

In the other case $U(i)<X$ for all the Δ (right) paths i separated by $(Z, Z+1)$. Then X can not be A and (X, m) is a terminal of a left path while $(X, 1)$ not. Hence $c(X-1, X)>c(Z, Z+1)=\Delta$, a contradiction. ▮

The proof of Theorem 1 is now complete. ∎

Remark. Observe that a non-boundary vertical lattice line can be deleted if no terminal point is on it since the column criterion continues to hold. So we can assume that $m \leqq k$, $n \leqq 2k$. Therefore the complexity of the algorithm depends only on k and not on m and n.

Theorem 1 does not answer the case when none of the four corners is exposed. In the first three examples in Figure 1 the reader can easily convince himself or herself, that no solution exists. But what simple reasons may prove the non-existence of the solution in the fourth example? Theorem 2 provides an answer. In order to incorporate Theorem 1, it will be convenient to consider the points $(0, m)$, $(n+1, m)$ as *fictitious exposed* points.

Theorem 2. *(a) If there is no (proper) exposed point on the upper line, the problem has a solution if and only if each path is trivial.*

(b) Assume that there is at least one proper exposed point.

b1. If $m < \Delta$, there is no solution.

b2. If $m > \Delta$, there always exists a solution.

b3. If $m = \Delta$, there exists a solution if and only if there are two exposed points (fictitious or not) on the upper line such that the congestion of any column between them is less than Δ.

(Observe that Theorem 1 is included. If $m = \Delta$ and A is exposed, the "column" between 0 and A is of zero congestion.)

Proof. Part (a) is trivial. We have seen b1. To see b2, let (X, m) be an exposed point. Apply a right-pushing on AX. Then we get a new problem on a rectangle of $m-1$ horizontal lines. But now the left upper corner became exposed and $m-1 \geqq \Delta$ so Theorem 1 applies.

b3. *Necessity.* Assume that the exposed points $X_0, X_1, \ldots, X_{t+1}$ $(t \geqq 1)$ on the upper line are separated by saturated columns (columns of congestion Δ) which are $(Z_1, Z_1+1), \ldots, (Z_{t+1}, Z_{t+1}+1)$. Each of the sets $\{1, 2, \ldots, Z_1\}$, $\{Z_1+1, \ldots, Z_2\}$, $\{Z_{t+1}+1, \ldots, n\}$ on the upper line has the property that the number of terminals in it and the number of edges leaving it have different parity. Therefore no solution uses every edge leaving such a set. But the edge which is not used cannot be (Z_i, Z_i+1) since each edge of a saturated column must be in a path. Hence these edges are vertical edges of the uppermost row $(m-1, m)$. This row contains n vertical edges. On the other hand, it must contain at least $t+2$ edges not in any path and $n-t$ edges which are in a path. This is impossible.

Suffiency. Suppose X and Y are two exposed points not separated by saturated columns. If Y is proper, first apply a left-pushing on YB then apply right-pushings on the maximal non-covered segments of YB. If X is proper, do the same on AX by changing the terms "left" and "right". We get therefore a problem in a rectangle of $m-1$ horizontal lattice lines in which the maximal congestion is $\Delta-1$. This latter follows from the proof of Theorem 1 applied to the segments AX and YB and from the hypothesis for the segment XY. Moreover, the left (right) corner became exposed if $X(Y)$ was proper. Apply Theorem 1. ∎

Return to the examples in Figure 1. The first and third do not satisfy condition (a). The second and fourth example do not satisfy b3. (See Figure 2.)

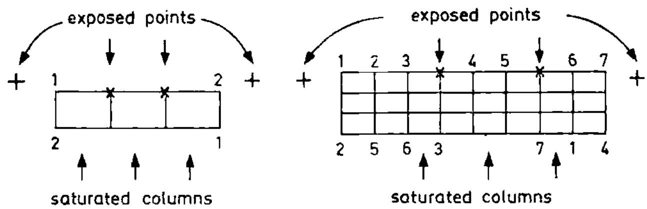

Fig. 2

3. The non-bipartite case

Henceforth we assume the terminals are positioned arbitrarily on the boundary of T. Furthermore, the corners are allowed to be assigned to two terminals. The two examples in Figure 3 may indicate some difficulties of finding a good characterization for this problem. One of them has a solution but the other has not.

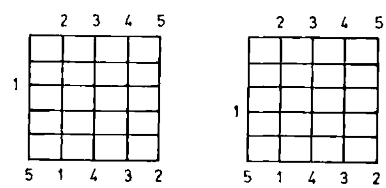

Fig. 3

We need the notion of odd sets. To this end, join each terminal pair by a new edge. A subset X of vertices of G is called *odd* if, in the extended graph, the number of edges (new or old) leaving X is odd. (In the sequel we do not need the new edges any longer.)

A simple parity argument shows that no solution uses every edge leaving X.

Call a cut of G *saturated* if its congestion is equal to the number of edges in it. Remark that columns and rows define cuts of special kind.

Let $\{r_1, r_2, \ldots, r_t\}$ be the set of saturated rows $(t \geqq 0)$ and let c be any column. These define $t+1$ disjoint sets T_i on the left-hand side of c. See Fig. 4.

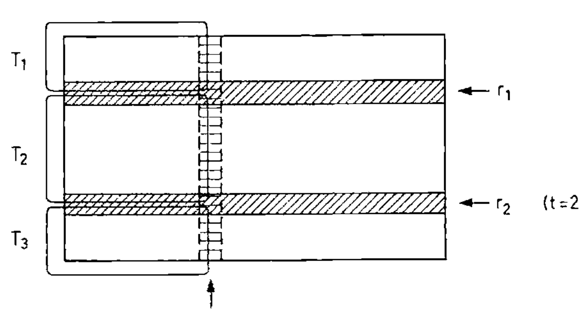

Fig. 4

The number of odd sets T_i is called the *parity congestion* of c.

The *revised congestion* or *r-congestion* of a column c is the sum of the parity congestion and the path congestion of c. (Note that the r-congestion would be the same if, in the definition of T_i, we had said "right-hand" instead of "left-hand".)

Revised column criterion. The r-congestion of any column is at most m.

The *revised row criterion* is defined in an analogous manner.

Theorem 3. *We are given a rectangle in a rectilinear grid and k pairs of terminals on its boundary. There exist k edge-disjoint paths between the corresponding terminals if and only if the revised row and column criteria hold.*

Remark. This theorem involves Theorem 2 but its proof provides a much less efficient (though polynomial-bounded) algorithm than that described in Section 2.

Proof. Necessity (of the revised column criterion). If T_i is odd, at least one edge e leaving T_i will not be used in any solution. Edge e is in column c, for othervise e would be in a saturated row which is impossible. Therefore the number m of edges in c should be at least the number of odd sets T_i plus the congestion of c, i.e. at least the r-congestion of c.

Sufficiency. The theorem is trivial for $m=1$, or $n=1$ so suppose that $m, n>1$. First we prove the theorem when there are no odd sets at all. In this case the congestion and r-congestion are the same. Instead of rectangles, we prove the assertion for n-rectangles. By a *near-rectangle* or *n-rectangle* we mean a region R bounded by the segments AY, YX, XB, BC, CD, DA where $A=(1, m-1)$, $Y=(v, m-1)$, $X=(v, m)$, $B=(n, m)$, $C=(n, 1)$, $D=(1, 1)$. Here $n>v\geqq 1$ and $m>2$. See Fig. 5.

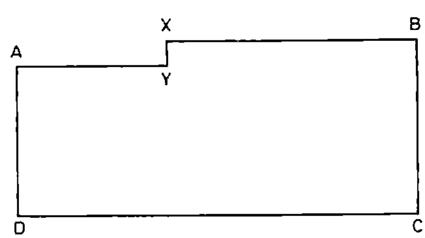

Fig. 5

The row (column) criterion is now slightly modified: the congestion of any row (column) r should be at most the number of edges in r.

The convex corners A, X, B, C, D may be assigned to either two or no terminals. The corner Y is assigned to no terminal and any other point of the boundary is assigned to exactly one terminal. We say that an n-rectangle is *satisfactory* if these assumptions and the row and column criteria hold.

Lemma. *In satisfactory n-rectangles there exist edge-disjoint points between the corresponding terminals.*

Proof. By induction on the area of R (which is integer). We are using a 'cutting off' operation at X which replaces R by R' defined by $X'=(v+1, m)$ $Y'=(v+1, m-1)$,

$A'=A$, $B'=B$, $C'=C$, $D'=D$ if $v<n-1$ and by $A'=A$, $B'=(m-1,n)$, $C'=C$, $D'=D$ if $v=n-1$. Moreover, a cutting off operation may change some terminals and introduce some new ones. Our general strategy is that we determine a cutting off in such a way that R' should be satisfactory and a solution in R' (that we have by the induction hypothesis) should imply a solution in R.

For a terminal pair i let $i(1)$ and $i(2)$ denote the two members of i.

For two points $V, W \neq X$ of the boundary we say that $W <' V$ if XWV is clockwise on the boundary.

Case 1. X is not exposed. Then two terminals are in X, say $i(1)=j(1)=X$. Suppose that $i(2) \leqq' j(2)$. Set $i(1):=X'$ and $j(1):=Y$ if $v<n-1$ and $i(1):=B'$, $j(1):=$ $:=(n-1,m-1)$ if $v=n$.

Claim. *R' is satisfactory.*

We prove the row and column criteria. The only row which can violate the row criterion is $(m-1,m)$. In this case $j(2)$ is on the segment $X'B$ and so is $i(2)$. Furthermore this row was saturated in R. But this is impossible.

The only column which can violate the column criterion is $c=(v, v+1)$. Then both $i(2)$ and $j(2)$ are on the left hand side of c and c was saturated in R. This is impossible if $v=1$. If $v>1$, the congestion of the column $(v-1, v)$ is at least that of $(v, v+1)$, that is the column $(v-1, v)$ was oversaturated in R, a contradiction. ▮

From a solution in R' we obtain one in R by extending the paths i and j with the edges (X, X') and (Y, X), respectively.

Case 2. X is exposed.

Subcase 2.1. Neither the column $(v, v+1)$ nor the row $(m-1,m)$ is saturated.

Cut off X and let i be a new terminal pair with $i(1)=X'$, $i(2)=Y$. Obviously R' is satisfactory and omitting the path i from the solution in R' we obtain a solution in R.

Subcase 2.2. The row $(m-1,m)$ is saturated.

Then there is no terminal pair j with both $j(1)$ and $j(2)$ on the segment XB. For notational convenience suppose that $j(2)$ is not on (X, B) for any j. Let $v<n-1$. Choose the terminal pair i in such a way that $i(1)$ is on XB and if $j(1)$ is on XB then $i(2) \geqq' j(2)$. Cut off X and replace the terminal pair i by i' and i'' where $i'(1)=i(1)$, $i'(2)=X'$, $i''(1)=Y$, $i''(2)=i(2)$.

If $v=n-1$ then $i(1)=j(1)=B$ for some i and j. Suppose that $i(2) \leqq' j(2)$. Cut off X and set $i(1):=(n, m-1)$, $j(1)=Y$.

Claim. *R' is satisfactory.*

Proof. The only danger may arise when X is less than the X-coordinate of $i(2)$ and there is a column c separating X and $i(2)$ such that c was saturated in R. But the choice of i implies that $j(2)$ is on the right-hand side of c whenever $j(1)$ is on XB. Thus the congestion of c may be at most $m-2$. ▮

From a solution in R' we obtain one in R by gluing the paths i', YXY', i''.

Subcase 2. The column $c=(v, v+1)$ is saturated but the row $(m-1, m)$ is not.

Suppose that $j(1)$ is on the right-hand side of c and $j(2)$ is on the other for all $j \in \mathcal{D}$, where $\mathcal{D}$ denotes the set of terminal pairs separated by c.

Choose $i \in \mathcal{D}$ in such a way that $j(1) \leqq' i(1)$ for any $j \in \mathcal{D}$.

Cut off X and replace i by i' and i'' where $i'(1)=i(1)$, $i'(2)=Y$, $i''(1)=X'$, $i''(2)=X'$. (The case $v=n-1$ is left to the reader.)

Claim. *R' is satisfactory.*

If U is a member of a terminal pair, denote the other by $t(U)$.

Proof. The only danger may arise when $i(1)$ is on CD and there is a row $r= =(\mu, \mu+1)$ above $i(1)$ and $i(2)$ which was saturated in R. By a simple induction we can prove that each column is saturated on the left-hand side of c and $t(k, 1)$ is on the right-hand side of c for $k=1, 2, \ldots, v$. Consequently the column $(1, 2)$ is saturated, there are two terminals in D and no terminal in C and no terminal pair j with $j(1), j(2)$ on CD. Using this, it can be shown that each row above r is saturated contradicting the assumption that $(m-1, m)$ is not saturated. ▌

From a solution in R' we obtain one in R by gluing the paths i', YXY', i''. The proof of the lemma is complete. ▌

Turning to the general case, observe that a set is odd if and only if it contains an odd number of odd vertices. Furthermore, in our case only the vertices on the boundary may be odd. What we are going to show is that there exists a pairing of the odd vertices such that the column and row criteria will hold if these pairs are considered as new terminal pairs to be connected. In this case the Lemma can be applied.

Suppose that the saturated rows and columns are $r_1, r_2, \ldots, r_t$ $(t \geqq 0)$ and $c_1, c_2, \ldots, c_s$ $(s \geqq 0)$, respectively. These rows and columns divide the graph into $(r+1) \cdot (s+1)$ disjoint parts P_i. It can be seen that each set P_i is even. Condiser one P_i in which the odd points are $X_1, X_2, \ldots, X_k$ (k is even) in this orde ron the boundary of T. Pair them in the natural way, that is, the pairs are $X_1X_2, X_3X_4, \ldots, X_{k-1}X_k$.

Claim. *The row and column criteria hold if these pairs are new terminal pairs to be connected.*

Proof. We prove the column criterion. Let c be any column. If c is saturated then it does not separate any new pair and thus c will satisfy the column criterion.

Let c be not saturated. Assume that c separates at most two new pairs. In this case c will violate the column criterion only if its congestion is one less than m and c separates exactly two new pairs. But this is impossible because of parity reasons.

If c separates more than two new pairs then $t>0$ and $s=0$.

Observe that in one P_i there may arise at most one new pair which is separated by c and one does arise if and only if the corresponding T_i (see Figure 6) is odd. This means that the number of new pairs separated by c is just equal to the parity congestion of c. Now the revised column criterion implies the claim. ▌

The proof of Theorem 3 is complete. ▌

Turn back to the examples in Figure 3. The first example has no solution since column c violates the revised column criterion. The second example has a solution. See Figure 7 (next page).

Remark. The proof of Theorem 3 involves an algorithm. In the first part we have to pair the odd vertices which is easy. The second part consists of the applications of about $m \cdot n$ cutting off operations. So the complexity is proportional to $m \cdot n$ comparing with the algorithm in Section 2 whose complexity depends only on the number of paths to be constructed.

Consequence. *If no saturated columns and rows exist, the problem always has a solution.*

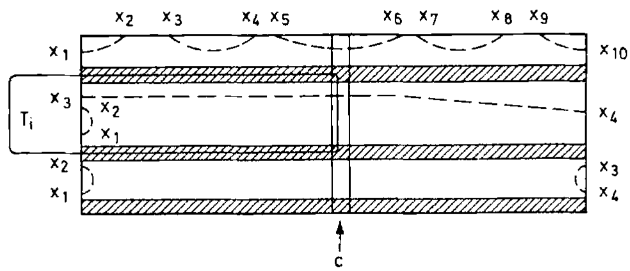

Fig. 6

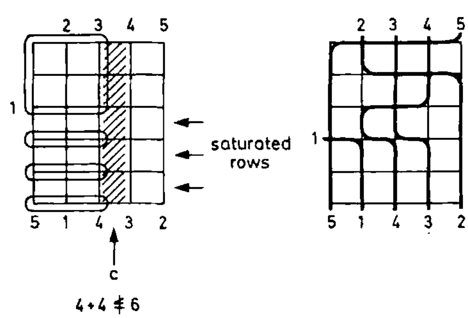

Fig. 7

4. Planar graphs

A related result of Okamura and Seymour [11] concerns the disjoint path problem for arbitrary planar graphs with a fixed embedding. Again, the terminals are specified on the boundary. The *cut criterion* requires that the congestion of any cut should not exceed the cardinality of the cut.

Theorem 4. (Cf. [12.]) *If there are no odd sets, the cut criterion is necessary and sufficient for the existence of edge disjoint paths between the corresponding terminals.* ∎

The Lemma, used in the proof of Theorem 3, can easily be derived from this theorem. Actually, the proof of the Lemma is an adaption of Seymour and Okamura's proof but it seemed to be worthwhile to work out the details here since the algorithm is significantly simpler for this special case.

It would be nice to find a common generalization of Theorem 3 and 4.

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